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# Recent topics on Multiplier Ideals

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## Contents



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# Introduction to Multiplier Ideals.

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- $(X, D)$ $X$ : normal variety, $D \geq 0$ : $\mathbb{Q}$-divisor on $X$
  ($D = \sum d_i D_i$, $d_i \in \mathbb{Q}_{\geq 0}$, $D_i \subset X$ : prime divisor)
  $K_X + D$ is $\mathbb{Q}$-cartier
  i.e. $^\exists r \in \mathbb{N}$ s.t. $r(K_X + D)$ is Cartier

- $(X, \mathfrak{a}^t)$ $X$ : $\mathbb{Q}$-Goren. normal, $\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$
  i.e. $\exists r \in \mathbb{N}$ s.t. $rK_X$ is cartier

- $(X, D; \mathfrak{a}^t)$ ...

Our main reference is Lazarsfeld's book [La2] for the theory of multiplier ideals.

log resol.

$f : \tilde{X} \to X$ log resol. of $D$ (resp. $\mathfrak{a}$)

$\overset{\text{def}}{\Longleftrightarrow}$
- $f$ : proper birat$^{l}$
- $\tilde{X}$ : smooth
- $\mathrm{Supp}(f_*^{-1}D) \cup \mathrm{Exc}(f)$ : SNC divisor
  ($f_*^{-1}D$: strict transform of $D$)

(resp. $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$ inv., $\mathrm{Supp}\, F \cup \mathrm{Exc}(f)$ is SNC.)

Def. of multiplier ideals

- $\mathcal{J}(D) = \mathcal{J}(X, D) := f_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner \underbrace{f^*(K_X + D)}_{\frac{1}{r} f^*(r(K_X + D))} \lrcorner) \subset \mathcal{O}_X$

- $\mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(X, \mathfrak{a}^t) := f_* \mathcal{O}_X(K_{\tilde{X}} - \llcorner (f^* K_X + tF) \lrcorner) \subset \mathcal{O}_X$

We can define $\mathcal{J}(X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_k^{t_k})$, $\mathcal{J}(X, D; \mathfrak{a}^t)$ similarly.

$(X, D)$ : klt (Kawamata log terminal) $\Leftrightarrow$ $\mathcal{J}(X, D) = \mathcal{O}_X$

$(X, D)$ : klt at $x$ $\Leftrightarrow$ $\mathcal{J}(X, D)_x = \mathcal{O}_{X,x}$

$X$ has only lt sing. at $x \in X$ $\Leftrightarrow$ $(X, 0)$ is klt at $x \in X$
(log terminal)

- Assume $X$ has only lt sing. at $x \in X$

$\mathrm{lct}(D; x) := \sup\{t \in \mathbb{Q} \mid \mathcal{J}(X, t \cdot D)_x = \mathcal{O}_{X,x}\}$

$\mathrm{lct}(\mathfrak{a}; x) := \sup\{t \in \mathbb{Q} \mid \mathcal{J}(X, \mathfrak{a}^t)_x = \mathcal{O}_{X,x}\}$

Basic properties

(1) $\mathcal{J}(D)$, $\mathcal{J}(\mathfrak{a}^t)$, etc. are indep. of the choice of the log resol. $f$. In particular,

$X$ : smooth
$\mathrm{Supp}(D)$ : SNC $\Rightarrow$ $\mathcal{J}(D) = \mathcal{O}_X(-\llcorner D \lrcorner)$

(2) $D_1 \geq D_2 \Rightarrow \mathcal{J}(D_1) \leq \mathcal{J}(D_2)$

$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \Rightarrow \mathcal{J}(\mathfrak{a}_1^t) \subseteq \mathcal{J}(\mathfrak{a}_2^t)$

Moreover if $\mathfrak{a}_2 \subseteq \overline{\mathfrak{a}_1} \Rightarrow \mathcal{J}(\mathfrak{a}_1^t) = \mathcal{J}(\mathfrak{a}_2^t)$

$\left( \begin{array}{l} g : Y \to X : \text{normalized blow-up along } \mathfrak{a} \text{ s.t. } \mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-E) \\ \overline{\mathfrak{a}} := g_* \mathcal{O}_Y(-E) \end{array} \right)$

(3) Assume $X$ has only lt sing.

$\Rightarrow \mathcal{J}(\mathfrak{a}) \supseteq \mathfrak{a}$

Moreover if $D$ is a cartier int. div. $\Rightarrow \mathcal{J}(D) = \mathcal{O}(-D)$

if $\mathfrak{a}$ is of pure ht 1 $\Rightarrow \mathcal{J}(\mathfrak{a}) = \mathfrak{a}$

( ☺ if $\mathfrak{a}$ is of pure ht 1 $\Rightarrow \mathfrak{a}$ is reflexive.

(4) $X$: $\mathbb{Q}$-Goren. affine var., $\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$

Fix $t < k \in \mathbb{N}$. Take general elements $x_1, \cdots, x_k \in \mathfrak{a}$

$A_i := \operatorname{div} x_i$, $D := \frac{1}{k} \Sigma A_i$

$\Rightarrow \mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(t \cdot D)$

(5) $\operatorname{lct}(D; x)$, $\operatorname{lct}(\mathfrak{a}; x) \in \mathbb{Q}_{>0}$

<u>Example</u>

(1) $X$: smooth var. of dim. $n$, $x \in X$, $\mathfrak{m} := \mathfrak{m}_{X,x}$

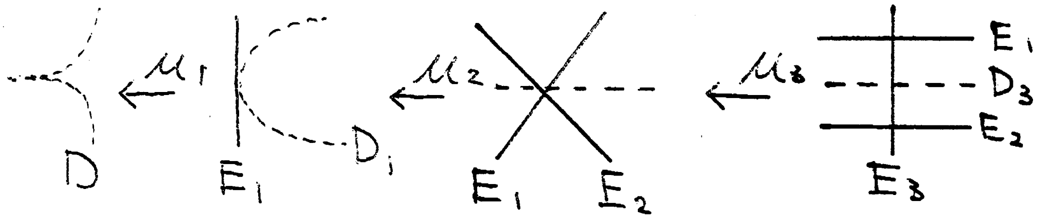
$\mathcal{J}(\mathfrak{m}^t) = \mathfrak{m}^{\lfloor t \rfloor + 1 - n}$ , $\mathrm{lct}(\mathfrak{m}; x) = n$

∵ $f : \hat{X} \to X$ : blow-up at $x$

$\mathfrak{m}\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(-E)$ , $K_{\hat{X}/X} = (n-1)E$

$\mathcal{J}(\mathfrak{m}^t) = f_*\mathcal{O}_{\hat{X}}((n-1-\lfloor t \rfloor)E) = \mathfrak{m}^{\lfloor t \rfloor + 1 - n}$

(2) $X = \mathbb{C}^2$, $D = (x^2 + y^3 = 0)$




$K_{X_1/X} = E_1$ , $K_{X_2/X} = E_1 + 2E_2$ , $K_{X_3/X} = E_1 + 2E_2 + 4E_3$

$f_1^* D = 2E_1 + D_1$, $f_2^* D = 2E_1 + 3E_2 + D_2$, $f_3^* D = 2E_1 + 3E_2 + 6E_3 + D_3$

$(f_i := \mu_i \circ \cdots \circ \mu_1 : X_i \to X)$

$\mathcal{J}(t \cdot D) = f_{3*}\mathcal{O}_{X_3}(\lceil 1-2t \rceil E_1 + \lceil 2-3t \rceil E_2 + \lceil 4-6t \rceil E_3 - \lfloor t \rfloor D_3)$

$\therefore \mathrm{lct}(D; 0) = \frac{5}{6}$ , $\mathcal{J}(\frac{5}{6} \cdot D) = (x, y)$

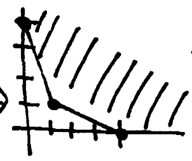
(3) $X = \mathrm{Spec}\, k[\sigma^\vee \cap M]$ : affine toric var.

$\mathfrak{a} \subseteq k[\sigma^\vee \cap M]$ : monomial ideal

$\mathfrak{a} \rightsquigarrow P(\mathfrak{a}) \subset M_{\mathbb{R}}$ : Newton polytope of $\mathfrak{a}$

i.e. convex hull of the set of exponents of the monomials in $\mathfrak{a}$

<u>ex</u>

$\mathfrak{a} = (x^4, xy, y^4) \subset \mathbb{C}[x, y] \Rightarrow$  $P(\mathfrak{a})$

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<u>Thm.</u> (Blickle [Bl], Hara-Yoshida [HY])

Assume $X = \operatorname{Spec} k[\sigma^\vee \cap M]$ is $\mathbb{Q}$-Goren. i.e. $^\exists r \in \mathbb{N}$ s.t. $rK_X$ : cartier

$^\exists u \in M$ s.t. $\operatorname{div} \chi^u = -rK_X$

$\omega := \frac{u}{r}$

$\Rightarrow \mathcal{J}(\mathfrak{a}^t) = \langle \chi^v \mid v + \omega \in \operatorname{Int}(t \cdot P(\mathfrak{a})) \subseteq M_{\mathbb{R}} \rangle$

<u>Cor.</u> (Howald [Ho1])

$X = \mathbb{C}^n$, $\mathfrak{a} \subseteq \mathbb{C}[x_1, \cdots, x_n]$ : monomial ideal

$\Rightarrow \mathcal{J}(\mathfrak{a}^t) = \langle \chi^v \mid v + \mathbb{1} \in \operatorname{Int}(t \cdot P(\mathfrak{a})) \subseteq \mathbb{R}^n \rangle$

<u>Proof</u>

$\mu : Y \to X$ : toric log resol. of $\mathfrak{a}$ s.t. $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-\underset{\sim}{F})$ torus inv.

$\rightsquigarrow \mathcal{J}(\mathfrak{a}^t)$ : monomial ideal

$v + \omega \in \operatorname{Int}(t \cdot P(\mathfrak{a}))$

$\Leftrightarrow v + \omega - \varepsilon v' \in t \cdot P(\mathfrak{a})$ $(0 < {}^\forall\varepsilon \ll 1, {}^\forall v' \in \operatorname{Int}(\sigma^\vee) \cap M)$

$\Leftrightarrow \mu^* \operatorname{div} \chi^v - \mu^* K_X - \varepsilon \mu^* \operatorname{div} \chi^{v'} \geq tF$

$\Leftrightarrow \mu^* \operatorname{div} \chi^v + K_Y - \underbrace{\ulcorner K_Y + \varepsilon \mu^* \operatorname{div} \chi^{v'} + \mu^* K_X + tF \urcorner}_{= \llcorner \mu^* K_X + tF \lrcorner} \geq 0$

$(\Leftrightarrow \chi^v \in \mathcal{J}(\mathfrak{a}^t))$

① $K_Y = -\sum D_i < 0$ ($D_i$'s are all the torus inv. div)

② $1 > (\text{coeff. of } \varepsilon \mu^* \operatorname{div} \chi^{v'}) \geq 0$

③ $\chi^{v'} \in \omega_X$ i.e. $\mu^* \operatorname{div} \chi^{v'}$ and $K_Y$ have the same support ∎

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$f \in \mathbb{C}[X] \rightsquigarrow \mathfrak{a}_f \subset \mathbb{C}[X]$: ideal gen. by the mono. appearing in $f$

$\mathcal{J}(t \cdot \mathrm{div}(f)) \subseteq \mathcal{J}(\mathfrak{a}_f^t)$

If $f$ is "general" $\Rightarrow$ $\mathcal{J}(t \cdot \mathrm{div}(f)) = \mathcal{J}(\mathfrak{a}_f^t)$ $(0 < {}^\forall t < 1)$

<u>Q.</u> How "general"?

<u>Thm.</u> (Howald [Ho2])

$f \in \mathbb{C}[X]$: non-degenerate.

$\left(\begin{array}{l} \text{i.e. } f \rightsquigarrow f_\sigma \quad \sigma: \text{face of } P(f) := P(\mathfrak{a}_f) \\ df_\sigma \text{ is nowhere vanishing on } (\mathbb{C}^*)^n \\ \text{for } {}^\forall \text{face } \sigma \text{ of } P(f) \end{array}\right.$

$\Rightarrow \mathcal{J}(t \cdot \mathrm{div}(f)) = \mathcal{J}(\mathfrak{a}_f^t)$, $0 < {}^\forall t < 1$

<u>Ex.</u>

$f = x_1^{d_1} + \cdots + x_n^{d_n}$ is non-degenerate. Assume $\sum \frac{1}{d_i} < 1$.

since $P(f) = \{(u_1, \cdots, u_n) \in \mathbb{R}^n \mid \sum u_i/d_i \geq 1\}$

$\mathrm{lct}(\mathrm{div}(f); 0) = \mathrm{lct}(\mathfrak{a}_f; 0) = \sum 1/d_i$

<u>Vanishing thm</u>

(i) (local vanishing)

$f: \tilde{X} \to X$: log resol. of $D$

(resp. $\mathfrak{a}$ s.t. $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$)

$\Rightarrow R^j f_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner f^*(K_X + D)\lrcorner) = 0$ , $(\forall j > 0)$

(resp. $R^j f_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner f^* K_X + tF \lrcorner) = 0$, $(\forall j > 0, \forall t > 0)$)

(ii) (Nadel vanishing)

$L$: cartier int. div. on $X$ s.t. $L-D$ is nef and big

$\Rightarrow H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0$ , $(\forall i > 0)$

<u>Cor.</u>

$X$: proj. normal var.

$B$: very ample divisor on $X$

$L$: cartier int. div. on $X$ s.t. $L-D$ is nef and big

$\Rightarrow \mathcal{O}_X(K_X + L + mB) \otimes \mathcal{J}(D)$ is gl. gen. if $m \geq \dim X$

$\because$ <u>Lem.</u> (Mumford [La1, Theorem 1.8.5])

$F$: coherent s.t. $H^i(X, F \otimes \mathcal{O}_X(-iB)) = 0$, $\forall i > 0$

$\Rightarrow F$ is gl. gen.

Nadel $\Rightarrow H^i(X, \mathcal{O}_X(K_X + L + (m-i)B) \otimes \mathcal{J}(D)) = 0$, $\forall i > 0$

$\Rightarrow$ O.K. ∎

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# Multiplier ideals and inversion of adjunction

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Recall adjunction formula
i.e.
$X$: smooth, $Y \subset X$: smooth divisor

$K_X + Y|_Y = K_Y$

$\overset{\text{generalize}}{\rightsquigarrow}$ $(X, S+B)$ $X$: normal var.
$S \subset X$: reduced divisor
$B \geq 0$: $\mathbb{Q}$-cartier on $X$
s.t. $S$ has no common comp. with $\mathrm{Supp}(B)$.
$K_X + S + B$ is $\mathbb{Q}$-cartier

$\nu : S^\nu \to S$ : normalization

$\exists B^\nu \geq 0$ $\mathbb{Q}$-divisor on $S^\nu$ (different of $B$ on $S^\nu$)

s.t. $K_{S^\nu} + B^\nu = \nu^*(K_X + S + B|_S)$

$$\begin{array}{ccccc} \tilde{X} & \supset & \tilde{S} & \xrightarrow{g} & S^\nu \\ f\downarrow & & \downarrow & \swarrow \nu & \\ X & \supset & S & & \end{array}$$

$f$: embedded resol.

$K_{\tilde{X}} + \tilde{S} \equiv f^*(K_X + S + B) + \sum a_i E_i$

$K_{\tilde{S}} \equiv g^*(K_{S^\nu} + B^\nu) + \sum a_i E_i|_{\tilde{S}}$

ex. $S$: normal cartier $\Rightarrow B^\vee = B|_S$

$$(X, S+B) \underset{\text{inv. of adj.}}{\overset{\text{adjunction}}{\rightleftarrows}} (S^\vee, B^\vee)$$

$d := \min\{a_i \mid f(E_i) \subset S\}$

$d_S := \min\{a_i \mid E_i|_{\hat{S}} \neq \phi\}$

Note

In general $d \leq d_S$

Thm

(i) (Kollár [K+], Shokurov [Sh])

$d > -1 \Leftrightarrow d_S > -1$

(ii) (Kawakita [Ka])

$d \geq -1 \Leftrightarrow d_S \geq -1$

Def.

(i) $(X, D, \mathfrak{a}^t)$ $X$: normal, $D \geq 0$: $\mathbb{Q}$-divisor on $X$, $\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$

$K_X + D$ is $\mathbb{Q}$-cartier,

$f: \tilde{X} \to X$: log resol. of $(D, \mathfrak{a})$, $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$

$\mathcal{J}(X, D, \mathfrak{a}^t) := f_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \llcorner f^*(K_X + D) + tF \lrcorner) \subset \mathcal{O}_X$

(ii) $(X,S;B,\mathfrak{a}^t)$ : $(X,S,B)$ as above, $\mathfrak{a}\subseteq\mathcal{O}_X$, $t>0$

$f:\tilde{X}\to X$ : log resol. of $(S+B,\mathfrak{a})$ s.t. $\tilde{S}:=f_*^{-1}S$ is smooth

$$\mathrm{adj}(X,S;B,\mathfrak{a}^t):=f_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}-\lfloor f^*(K_X+S+B)+tF\rfloor+\tilde{S})\subset\mathcal{O}_X$$

$$\cup$$

$$\mathcal{J}(X,S+B,\mathfrak{a}^t)$$

Remark

(0) $B=0$, $\mathfrak{a}=\mathcal{O}_X \Rightarrow \mathrm{adj}(X,S):=\mathrm{adj}(X,S;B,\mathfrak{a}^t)$

(i) $\mathrm{adj}(X,S;B,\mathfrak{a}^t)$ is indep. of the choice of $f$.

(ii) $X$ : $\mathbb{Q}$-Goren. affine, $\mathrm{ht}\,\mathfrak{a}\geq 2$, $f\in\mathfrak{a}$ is general

$\Rightarrow \mathcal{J}(X,\mathfrak{a})=\mathrm{adj}(X,\mathrm{div}(f))$

Note

- $d_S>-1 \iff \mathcal{J}(S^\nu,B^\nu)=\mathcal{O}_{S^\nu}$

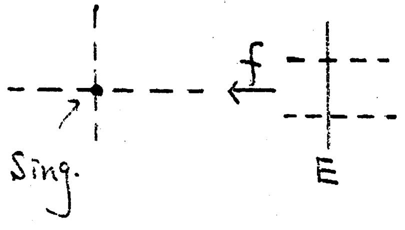
  $\iff (S^\nu,B^\nu)$ : klt

- $d>-1 \iff \mathrm{adj}(X,S;B)=\mathcal{O}_X$ near $S$

  $\iff (X,S+B)$ : plt near $S$
  (purely log terminal)

Ex

(i) $X=\mathbb{C}^2$, $S=(xy=0)$, $B=0$, $\mathfrak{a}=\mathcal{O}_X$




$K_{\tilde{X}/X}=E$, $f^*S=\tilde{S}+2E$

$\mathrm{adj}(X,S)=f_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}-f^*K_X-f^*S+\tilde{S})$

$=f_*\mathcal{O}_{\tilde{X}}(-E)=(x,y)$

(ii) $X$: $\mathbb{Q}$-Goren. normal surface, $S \subset X$: reduced cartier divisor.

$\Rightarrow \mathrm{adj}(X,S)\mathcal{O}_S = C(S) := \mathrm{Ann}(\nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S)$

$$\left(\begin{array}{l} \nu: S^\nu \to S : \text{normalization} \\ \quad \tilde{S} \cap \tilde{X} \to \cap X : \text{embedded resol} \end{array}\right)$$

Thm (Restriction thm, c.f. [La2, Theorem 9.5.1])

$(X, S; B, \mathfrak{a}^t)$ as above, Assume $\mathfrak{a} \not\subset I_S = \mathcal{O}_X(-S)$

$\Rightarrow \mathrm{adj}(X,S;B,\mathfrak{a}^t)\mathcal{O}_S = \nu_*\mathcal{J}(S^\nu, B^\nu, \mathfrak{a}\mathcal{O}_{S^\nu}) \subset \mathcal{O}_S$

In particular

$d > -1 \Leftrightarrow d_S > -1$ in this case $S^\nu \cong S$ (i.e. $S$: normal)

proof

For simplicity assume $\mathfrak{a} = \mathcal{O}_X$

$$\begin{array}{ccccc} \tilde{X} & \supset & \tilde{S} & \xrightarrow{g} & \\ f\downarrow & & \downarrow & & S^\nu \\ X & \supset & S & \xleftarrow{\nu} & \end{array} \qquad f: \text{embedded resol}$$

$$0 \to \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \lfloor f^*(K_X+S+B)\rfloor) \to \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \lfloor f^*(K_X+S+B)\rfloor + \tilde{S})$$
$$\xrightarrow{\cdot\mathcal{O}_{\tilde{S}}} \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \lfloor g^*(K_{S^\nu}+B^\nu)\rfloor) \to 0$$

$\downarrow f_*$

$$0 \to \mathcal{J}(X, S+B) \to \mathrm{adj}(X,S;B) \xrightarrow{\cdot\mathcal{O}_S} \nu^*\mathcal{J}(S^\nu, B^\nu)$$
$$\to R^1 f_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} - \lfloor f^*(K_X+S+B)\rfloor) \underset{\uparrow}{=} 0$$

local vanishing

■

<u>proof of Thm (ii)</u> (Kawakita)

The question is local $\rightsquigarrow$ discuss over a germ at a closed pt. $x \in S \subset X$ ($X = \operatorname{Spec} R$, $R$: local)

Assume $d_S \geq -1$ ($\Leftrightarrow$ $(S^\nu, B^\nu)$ is lc)

$\mathfrak{a}_0 := \operatorname{adj}(X, S; B) \subset \mathcal{O}_X$

$\mathfrak{a}_{n+1} := \operatorname{adj}(X, S; B, \mathfrak{a}_n^{1-\varepsilon_n})$, $0 < \varepsilon_n \ll 1$

$\mathfrak{a}_1 \mathcal{O}_S = \operatorname{adj}(X, S; B, \mathfrak{a}_0^{1-\varepsilon})\mathcal{O}_S = \nu_* \mathcal{J}(S^\nu, B^\nu, \mathfrak{a}_0 \mathcal{O}_S^{1-\varepsilon}) \supset \mathfrak{a}_0 \mathcal{O}_S$

($\because$ $d_S \geq -1$, $\mathfrak{a}_0 \mathcal{O}_S = \nu_* \mathcal{J}(S^\nu, B^\nu)$)

since $\mathfrak{a}_0 \supset \mathfrak{a}_1$,

$\mathfrak{a}_1 \mathcal{O}_S = \mathfrak{a}_0 \mathcal{O}_S$, i.e. $\mathfrak{a}_1 + I_S = \mathfrak{a}_0 + I_S$

Thus $\begin{cases} \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \\ \mathfrak{a}_0 + I_S = \mathfrak{a}_1 + I_S = \mathfrak{a}_2 + I_S = \cdots \end{cases}$

Suppose $d < -1$, $\quad (K_{\tilde{X}} + \tilde{S} \equiv f^*(K_X + S + B) + \Sigma a_j E_j)$

$\Rightarrow \exists E_i$ : $f$-exc. divisor on $\tilde{X}$ s.t. $a_i < -1$.

$\mathfrak{a}_0 = \operatorname{adj}(X, S; B) \subset f_* \mathcal{O}_{\tilde{X}}(\ulcorner a_i \urcorner E_i) = f_* \mathcal{O}_{\tilde{X}}(-E_i)$

$$\begin{aligned} \mathfrak{a}_1 = \operatorname{adj}(X, S; B, \mathfrak{a}_0^{1-\varepsilon}) &\subset \operatorname{adj}(X, S; B, f_* \mathcal{O}_{\tilde{X}}(-E_i)^{1-\varepsilon}) \\ &\subset f_* \mathcal{O}_{\tilde{X}}(\ulcorner a_i - (1-\varepsilon) \urcorner E_i) \\ &= f_* \mathcal{O}_{\tilde{X}}(-2E_i) \\ &(\because \ \varepsilon \ll 1) \end{aligned}$$

$\therefore \mathfrak{a}_n \subset f_* \mathcal{O}_{\tilde{X}}(-(n+1)E_i)$, $\forall n \geq 0$

On the other hand, by Nagata's thm,

$\forall \ell \in \mathbb{N}$, $^{\exists} k(\ell) \in \mathbb{N}$ s.t. $f_* \mathcal{O}_{\tilde{X}}(-k(\ell)E) \subset m_{x,X}^{\ell}$

$\therefore \mathfrak{a}_0 \subset \bigcap_{n \in \mathbb{N}} (\mathfrak{a}_n + I_S) \subset \bigcap_{\ell \in \mathbb{N}} (m_{x,X}^{\ell} + I_S) = I_S$

this implies $\nu_* \mathcal{J}(S^\nu, B^\nu) = 0$ contradiction.

$\therefore d \geq -1$ ∎

<u>Conj (Kollár, Shokurov)</u> (See [K+] and [Sh])

$^\forall Z \subset S$ : closed subset

$d(Z) := \min\{a_i \mid f(E_i) \subset Z\}$

$d_S(Z) := \min\{a_i \mid E_i|_{\tilde{S}} \neq \phi,\ f(E_i|_{\tilde{S}}) \subset Z\}$

$d(Z) = d_S(Z)$ ?

( $\leq$ o.K. )

<u>known case</u> (Ein-Mustaţă [EM], cf. [EMY])

$X$ : l.c.i , $S$ : normal cartier

<u>Higher codimension</u>

$X$ : $\mathbb{Q}$-Goren. normal var./$\mathbb{C}$

$Y := \sum^{k} t_i Y_i$ , $t_i > 0$ , $Y_i \subsetneq X$ : closed subscheme

$\mathfrak{a}_i \subseteq \mathcal{O}_X$ : def. ideal of $Y_i$

$f : \tilde{X} \to X$ : log resol. of $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$

$\mathfrak{a}_i \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F_i)$

$K_{\tilde{X}/X} - \Sigma t_i F_i \equiv \Sigma a_j E_j$

$(X, Y)$: $\overset{p}{k}lt \iff a_j > -1, \ \forall j$

$(X, Y)$: $lc \iff a_j \geq -1, \ \forall j$

<u>Thm</u> (T- [Ta1])

$(X, Y)$ as above. Assume $X$ is smooth

$Z \subsetneq X$: $\mathbb{Q}$-Goren. closed subvar. s.t. $Z \not\subset \cup Y_i$

$(Z, Y|_Z)$: $lc \Rightarrow (X, Y+Z)$: $lc$ near $Z$

<u>proof</u>

For simplicity, assume $Y = 0$

$L \subset Z$: locus of $lc$ sing.

i.e. $L$ is defined by $\mathcal{J}(Z) = \mathcal{J}(Z, \mathcal{O}_Z) \subset \mathcal{O}_Z$

(since $Z$ is $lc$, $L$ is reduced)

$(I_Z \subset) I_L \subset \mathcal{O}_X$: def. ideal of $L$ in $X$

i.e. the lift of $\mathcal{J}(Z)$

We have the following two restriction thm.

① $\mathcal{J}(Z, (\mathfrak{a}\mathcal{O}_Z)^t) \subset I_L \mathcal{J}(X, \mathfrak{a}^t)\mathcal{O}_Z$, $\forall \mathfrak{a} \subset \mathcal{O}_X$, $\forall t > 0$

② $\mathcal{J}(Z, (\mathfrak{a}\mathcal{O}_Z)^t) \subset \mathcal{J}(X, \mathfrak{a}^t I_Z^{1-\varepsilon})\mathcal{O}_Z$, $\forall \mathfrak{a} \subset \mathcal{O}_X$, $\forall t > 0$, $0 < \forall \varepsilon \ll 1$

(We prove these by char. $p > 0$ method, later)

$$Z : \mathrm{lc} \Rightarrow \underset{\mathcal{J}(Z)}{\mathcal{J}(Z, (\underset{\|}{I_L \mathcal{O}_Z})^{1-\varepsilon})} \supset \underset{\mathcal{J}(Z)}{\underset{\|}{I_L \mathcal{O}_Z}}, \quad 0 < {}^{\forall}\varepsilon \ll 1$$

by ①, $I_L \mathcal{O}_Z \subset \mathcal{J}(Z, (I_L \mathcal{O}_Z)^{1-\varepsilon}) \underset{①}{\subseteq} I_L \mathcal{J}(X, I_L^{1-\varepsilon}) \mathcal{O}_Z$

$\Rightarrow \mathcal{J}(X, I_L^{1-\varepsilon}) = \mathcal{O}_X$

by ②, $I_L \mathcal{O}_Z \subset \mathcal{J}(Z, (I_L \mathcal{O}_Z)^{1-\varepsilon}) \overset{②}{\subset} \mathcal{J}(X, I_L^{1-\varepsilon} I_Z^{1-\varepsilon}) \mathcal{O}_Z$

since $\mathcal{J}(X, I_L^{1-\varepsilon} I_Z^{1-\varepsilon}) \supset I_Z \mathcal{J}(X, I_L^{1-\varepsilon}) = I_Z$,

$I_L \subset \mathcal{J}(X, I_L^{1-\varepsilon} I_Z^{1-\varepsilon}), \quad 0 < {}^{\forall}\varepsilon \ll 1$

$\rightsquigarrow (X, Z) : \mathrm{lc}$ near $Z$ ▨

<u>Sketch of proof ②</u> (we can prove ① similarly)

Assume $X = \operatorname{Spec} R$ ($(R, \mathfrak{m})$: complete RLR of char.0)

$Z = \operatorname{Spec} S$ ($S = R/I$, $I = \sqrt{I} \subset R$ : unmixed) → char.p>0

<u>ETS</u> $\tau(S, (\mathfrak{a}S)^t) \subset \tau(R, \mathfrak{a}^t I^{1-\varepsilon}) S$

${}^{\forall}\mathfrak{a} \subseteq R$, ${}^{\forall}t > 0$, $0 < {}^{\forall}\varepsilon \ll 1$

dual

$$0_{E_S}^{*(\mathfrak{a}S)^t} \supset \underset{Z}{\underset{\Downarrow}{0_{E_R}^{*\mathfrak{a}^t I^{1-\varepsilon}}}} \cap E_S$$

$E_S := E_S(S/\mathfrak{m}S)$, $E_R := E_R(R/\mathfrak{m})$,

$E_S \cong (0 : I)_{E_R} \subset E_R$

$\mathfrak{a}^{\lceil tq \rceil} I^{\lceil q(1-\varepsilon) \rceil} F_R^e(z) = 0 \in \mathbb{F}_R^e(E_R) \cong E_R \quad {}^{\forall} q = p^e \gg 0$

$F_R^e : E_R \to \mathbb{F}_R^e(E_R) \cong E_R$

$F_S^e : E_S \to \mathbb{F}_S^e(E_S)$

$$(q = p^e)$$

$\forall z \in E_S$, $F_S^e(z) = 0 \in \mathbb{F}_S^e(E_S) \Leftrightarrow (I^{[q]} : I) F_R^e(z) = 0 \in E_R$

since $I^{[q]} : I \subset I^{q-1} \subset I^{\lceil q(1-\varepsilon) \rceil}$, $q = p^e \gg 0$, $0 < {}^\forall\varepsilon \ll 1$

$\mathfrak{a}^{\lceil tq \rceil} (I^{[q]} : I) F_R^e(z) = 0$, $q = p^e \gg 0$,

$\Leftrightarrow (\mathfrak{a}S)^{\lceil tq \rceil} F_S^e(z) = 0$, $q = p^e \gg 0$

$\Rightarrow z \in 0_{E_S}^{*(\mathfrak{a}S)^t}$ $\blacksquare$

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A char. p analog of adjoint ideals (Appendix)

$(R, \mathfrak{m})$ : F-finite normal local of char. $p>0$

$f \neq 0$, $\mathfrak{a} \subset R$, $t > 0$

$\tau^{\mathrm{div}}(R, f; \mathfrak{a}^t) := \mathrm{Ann}_R\, 0_E^{*(f;\mathfrak{a}^t)}$

$E := E_R(R/\mathfrak{m}) \cong H^d_{\mathfrak{m}}(\omega_R)$

$z \in 0_E^{*(f;\mathfrak{a}^t)} \overset{\text{def}}{\Longleftrightarrow} {}^{\exists} c \underset{R}{\ni} \notin {}^{\forall}$min. prime of $R/f$

s.t. $c f^{q-1} \mathfrak{a}^{\lceil tq \rceil} z^q = 0$, $q = p^e \gg 0$

$\left( \begin{array}{l} F^e : E \to \mathbb{F}^e(E) := {}^eR \otimes_R E \\ \quad z \mapsto z^q := 1 \otimes z \end{array} \right.$

If $R$ is $\mathbb{Q}$-Goren.

$R/f$ is $\mathbb{Q}$-Goren, normal

$\Rightarrow \tau(R/f, (\mathfrak{a} R/f)^t) = \tau^{\mathrm{div}}(R, f; \mathfrak{a}^t) R/f$

See [Ta3] for details

Ex

$R = k[[X, Y]]$, $f = XY$, $\mathfrak{a} = R$

$\Rightarrow \tau^{\mathrm{div}}(R, f) = (X, Y)$

$\quad\quad \parallel$

$\quad\quad \tau^{\mathrm{div}}(R, f; R)$

Thm (T- [Ta3])

$(R, \mathfrak{m})$ : normal local ring ess. of finite type $/\mathbb{C}$

$f \neq 0 \in R$, $\mathfrak{a} \subset R$, $t > 0$

$(\widetilde{R}, \widetilde{f}, \widetilde{\mathfrak{a}})$ : reduction to char. $p \gg 0$ of $(R, f, \mathfrak{a})$

$\Rightarrow \widetilde{\mathrm{adj}(\mathrm{Spec} R, \mathrm{div}(f); \mathfrak{a}^t)} = \tau^{\mathrm{div}}(\widetilde{R}, \widetilde{f}; \widetilde{\mathfrak{a}}^t)$

# Applications of asymptotic multiplier ideals

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local properties of multiplier ideals

(1) (Restriction thm)

$X$: normal $\mathbb{Q}$-Goren. var./$\mathbb{C}$

$S \subset X$: normal $\mathbb{Q}$-Goren. cartier divisor

$\mathfrak{a} \subseteq \mathcal{O}_X$, $t > 0$. Assume $S \not\subset \mathrm{Zeroes}(\mathfrak{a})$.

$\Rightarrow \mathcal{J}(S, (\mathfrak{a}\mathcal{O}_S)^t) = \mathrm{adj}(X, S; \mathfrak{a}^t)\mathcal{O}_S \subset \mathcal{J}(X, \mathfrak{a}^t)\mathcal{O}_S$

(2) (Subadditivity) (Demailly-Ein-Lazarsfeld [DEL])

$X$: smooth

$\Rightarrow \mathcal{J}(\mathfrak{a}^s \mathfrak{b}^t) \subset \mathcal{J}(\mathfrak{a}^s)\mathcal{J}(\mathfrak{b}^t)$, $\forall \mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_X$, $\forall s, t > 0$

(More generally

$x \in X$, $\mathcal{J}(\mathfrak{a}^s \mathfrak{b}^t)_x \subset \sum_{\substack{\lambda+\mu = \dim X \\ \lambda, \mu \geq 0}} \mathcal{J}(\mathfrak{a}^s \mathfrak{m}_{X,x}^{\lambda})_x \, \mathcal{J}(\mathfrak{b}^t \mathfrak{m}_{X,x}^{\mu})_x$

$\subseteq \mathcal{J}(\mathfrak{a}^s)_x \, \mathcal{J}(\mathfrak{b}^t)_x$ )

(3) (Summation) (Mustaţă [Mu])

$X$ : smooth

$$\Rightarrow \mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^t) \subset \sum_{\substack{\lambda+\mu=t \\ \lambda,\mu\geq 0}} \mathcal{J}(X, \mathfrak{a}^\lambda)\, \mathcal{J}(X, \mathfrak{b}^\mu)$$

In particular

$$\mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^{s+t}) \subset \mathcal{J}(X, \mathfrak{a}^s) + \mathcal{J}(X, \mathfrak{b}^t)$$

<u>sketch of proof (2)</u>

$$\begin{tikzcd} X \cong \Delta \arrow[r, hook] & X\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

since $X$ : smooth, $\Delta \hookrightarrow X\times X$ c.I. (diagonal embedding)

$$\begin{tikzcd} \widetilde{X}_1 \arrow[d] & \widetilde{X}_1\times\widetilde{X}_2 \arrow[l] \arrow[r] \arrow[d] & \widetilde{X}_2 \arrow[d] \\ X_1 & X_1\times X_2 \arrow[l] \arrow[r] & X_2 \end{tikzcd}$$

$$\mathcal{J}(X, \mathfrak{a}^s\mathfrak{b}^t) \subset \mathcal{J}(X\times X, (p_1^{-1}\mathfrak{a})^s(p_2^{-1}\mathfrak{b})^t)\mathcal{O}_\Delta$$

repeated applications of Restriction thm

$$\mathcal{J}(X\times X, (p_1^{-1}\mathfrak{a})^s(p_2^{-1}\mathfrak{b})^t) = p_1^{-1}\mathcal{J}(X, \mathfrak{a}^s)\cdot p_2^{-1}\mathcal{J}(X, \mathfrak{b}^t)$$

$$\Rightarrow \mathcal{J}(X, \mathfrak{a}^s\mathfrak{b}^t) \subset \mathcal{J}(X, \mathfrak{a}^s)\,\mathcal{J}(X, \mathfrak{b}^t) \quad \blacksquare$$

<u>Ex</u> (cf. [TW])

$X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^5)$ $A_4$-sing.

$\mathfrak{a} = (x, y^4, y^3z, y^2z^2, yz^3, z^4)$

$\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$, $\mathcal{J}(\mathfrak{a}^{\frac{1}{2}}) = (x, y^2, yz, z^2)$

$$\rightsquigarrow \underset{\mathfrak{a}^{\frac12}\mathfrak{a}^{\frac12}}{\overset{\parallel}{\mathcal{J}(\mathfrak{a})}} \not\subset \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^2$$

$x \in \mathcal{J}(\mathfrak{a})$, $x \notin \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^2$

Sing. case (T- [Ta 2])

(2)$'$ (Subadditivity)

$X$ : $\mathbb{Q}$-Goren. normal var./$\mathbb{C}$

$\Rightarrow J \cdot \mathcal{J}(\mathfrak{a}^s\mathfrak{b}^t) \subset \mathcal{J}(\mathfrak{a}^s)\mathcal{J}(\mathfrak{b}^t)$, $\forall \mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$, $\forall s, t > 0$

$\left(\begin{array}{l} J \subseteq \mathcal{O}_X : \text{Jacobian ideal sheaf} \\ \text{We cannot replace } J \text{ by } \sqrt{J} \end{array}\right)$

(3)$'$ (Summation)

$X$ : $\mathbb{Q}$-Goren. normal var.

$$\Rightarrow \mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^t) = \sum_{\substack{\lambda+\mu=t \\ \lambda,\mu\geq 0}} \mathcal{J}(X, \mathfrak{a}^\lambda\mathfrak{b}^\mu)$$

In particular

$$J \cdot \mathcal{J}(X, (\mathfrak{a}+\mathfrak{b})^t) \subset \sum_{\substack{\lambda+\mu=t \\ \lambda,\mu\geq 0}} \mathcal{J}(X, \mathfrak{a}^\lambda)\mathcal{J}(X, \mathfrak{b}^\mu)$$

($J$ : Jacobian)

Sketch of proof (2)$'$

Assume $X = \operatorname{Spec} R$, $R$ : complete local of char. 0

$\rightsquigarrow$ char. $p > 0$

ETS $J \cdot \tau(\mathfrak{a}^s\mathfrak{b}^t) \subset \tau(\mathfrak{a}^s)\tau(\mathfrak{b}^t)$

dual $\qquad \tau(\mathfrak{b}^t) := \operatorname{Ann} 0_E^{*\mathfrak{b}^t}$, $E := E_R(R/\mathfrak{m})$

$$(0_E^{*\mathfrak{a}^s\mathfrak{b}^t} : J)_E \supset (0_E^{*\mathfrak{b}^t} : \tau(\mathfrak{a}^s))_E$$

$\qquad\qquad\qquad z \qquad\qquad\qquad\qquad \overset{\Downarrow}{z}$

$\rightsquigarrow \tau(\mathfrak{a}^s)z \in 0_E^{*\mathfrak{b}^t}$

i.e. $\exists\, c \in R^\circ$ s.t. $c\mathfrak{b}^{\lceil tq \rceil}\tau(\mathfrak{a}^s)^{[q]}z^q = 0 \in F^e(E)$,

$(R^\circ := R \setminus \bigcup_{P:\text{ minimal prime}} P)$ $\quad \forall q = p^e \gg 0$

<u>claim</u> $\exists\, d \in R^\circ$ s.t. $d\mathfrak{a}^{\lceil sq \rceil}J^{[q]} \subset \tau(\mathfrak{a}^s)^{[q]}$, $\forall q = p^e \gg 0$.

If we accept this claim

$\Rightarrow cd\mathfrak{a}^{\lceil sq \rceil}\mathfrak{b}^{\lceil tq \rceil}J^{[q]}z^q = 0 \in F^e(E)$ $\forall q = p^e \gg 0$

$\Rightarrow Jz \subset 0_E^{*\mathfrak{a}^s\mathfrak{b}^t}$ ▨

<u>Ex.</u>

$X, \mathfrak{a}$ as above Ex.

$J = (x, y, z^4)$ , $\sqrt{J} = (x, y, z)$

$(x, y, z^4)\,\mathcal{J}(\mathfrak{a}) \subset \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^{\frac{1}{2}}$

$(x, y, z)\,\mathcal{J}(\mathfrak{a}) \not\subset \mathcal{J}(\mathfrak{a}^{\frac{1}{2}})^{\frac{1}{2}}$

$\underset{xz}{\underset{\cup}{\phantom{x}}}$ $\not\in$

<u>Asymtopic multiplier ideals</u> (See [ELS] or [La2] for details)

$X$: $\mathbb{Q}$-Goren. normal var.

$\mathfrak{a}_\bullet$ : graded family of ideals on $X$

i.e. $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbb{N}}$

$\mathfrak{a}_0 = \mathcal{O}_X$, $\mathfrak{a}_1 \neq 0$, $\mathfrak{a}_m \subset \mathcal{O}_X$

$\mathfrak{a}_k \cdot \mathfrak{a}_l \subset \mathfrak{a}_{k+l}$ , $\forall k, l \in \mathbb{N}$

$t>0$ fix.

$\mathcal{J}(\mathfrak{a}_{k\ell}^{\frac{t}{k\ell}}) \supset \mathcal{J}(((\mathfrak{a}_k)^{\ell})^{\frac{t}{k\ell}}) = \mathcal{J}(\mathfrak{a}_k^{\frac{t}{k}})$

$\rightsquigarrow \{\mathcal{J}(\mathfrak{a}_m^{\frac{t}{m}})\}_{m\in\mathbb{N}}$ has a unique max. element w.r.t. inclusion.

Denote it by $\mathcal{J}(\mathfrak{a}_\bullet^t)$

Ex.

(1) $\mathfrak{a}_m := \mathfrak{a}^m$, $\mathfrak{a} \subseteq \mathcal{O}_X$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\bullet^t) = \mathcal{J}(\mathfrak{a}^t)$

(2) $L$: linear system,

$\mathfrak{a}_m := \mathfrak{b}(|mL|)$ : base ideal of $|mL|$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\bullet^t) =: \mathcal{J}(t\cdot\|L\|)$

(3) $X = \operatorname{Spec} R$, $\mathfrak{p} \subset R$: prime ideal

$\mathfrak{a}_m := \mathfrak{p}^{(m)} := \mathfrak{p}^m R_{\mathfrak{p}} \cap R$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\bullet^t) =: \mathcal{J}(t\cdot\mathfrak{p}^{(\bullet)})$

Basic properties

(1). $t_1 > t_2 \Rightarrow \mathcal{J}(\mathfrak{a}_\bullet^{t_1}) \subset \mathcal{J}(\mathfrak{a}_\bullet^{t_2})$

(2). $\mathfrak{a}_\bullet$, $\mathfrak{b}_\bullet$,

If $0 \neq {}^\exists c \subset \mathcal{O}_X$ s.t. $c\,\mathfrak{a}_m \subset \mathfrak{b}_m$, ${}^\forall m \gg 0$

$\Rightarrow \mathcal{J}(\mathfrak{a}_\bullet^t) \subset \mathcal{J}(\mathfrak{b}_\bullet^t)$

(3). $\mathfrak{a}_k \mathcal{J}(\mathfrak{a}_\bullet^l) \subset \mathcal{J}(\mathfrak{a}_\bullet^{k+l})$ , $k, l \in \mathbb{N}$

In particular, if $X$ has only lt sing.

$(\Leftrightarrow \mathcal{J}(\mathcal{O}_X) = \mathcal{O}_X)$

$\Rightarrow \mathfrak{a}_k \subset \mathcal{J}(\mathfrak{a}_\bullet^k)$ , $\forall k \in \mathbb{N}$

(4).(Restriction) $S$: normal $\mathbb{Q}$-Goren Cartier divisor on $X$

$\mathcal{J}(S, (\mathfrak{a}_\bullet \mathcal{O}_S)^t) \subset \mathcal{J}(X, \mathfrak{a}_\bullet^t)\mathcal{O}_S$ $(\forall t > 0)$

(5).(Subadditivity)

$J \subset \mathcal{O}_X$ : Jacobian ideal

$J^{l-1}\mathcal{J}(\mathfrak{a}_\bullet^{kl}) \subset \mathcal{J}(\mathfrak{a}_\bullet^k)^l$ , $k, l \in \mathbb{N}$

(6).(Summation)

$(\mathfrak{a}_\bullet + \mathfrak{b}_\bullet)_m := \sum_{k+l=m} \mathfrak{a}_k \cdot \mathfrak{b}_l$

$\mathcal{J}((\mathfrak{a}_\bullet + \mathfrak{b}_\bullet)^t) \subset \sum_{\lambda+\mu=t} \mathcal{J}(\mathfrak{a}_\bullet^\lambda \mathfrak{b}_\bullet^\mu)$

(7). $0 \neq \exists \mathfrak{c} \subset \mathcal{O}_X$ s.t. $\mathfrak{c}\,\mathfrak{a}_m \subset \mathcal{J}(\mathfrak{b}_\bullet^m)$ , $\forall m \gg 0$

$\Rightarrow J \cdot \mathfrak{a}_m \subset \mathcal{J}(\mathfrak{b}_\bullet^m)$ , $\forall m \in \mathbb{N}$

( $J \subset \mathcal{O}_X$ : Jacobian ideal )

<u>short proof</u>

(2). $\mathcal{J}(\mathfrak{a}_\bullet^t) = \mathcal{J}(\mathfrak{a}_m^{\frac{t}{m}}) = \mathcal{J}(\mathfrak{c}^{\frac{t}{m}} \mathfrak{a}_m^{\frac{t}{m}})$

$\subset \mathcal{J}(\mathfrak{b}_m^{\frac{t}{m}}) \subset \mathcal{J}(\mathfrak{b}_\bullet^t)$ $m \gg 0$

(7). $\mathfrak{c} J^l \mathfrak{a}_m^l \subset \mathfrak{c} J^l \mathfrak{a}_{ml}$ use subadditivity

$\subset J^l \mathcal{J}(\mathfrak{b}_\bullet^{ml}) \subset \mathcal{J}(\mathfrak{b}_\bullet^m)^l$ , $l \gg 0$

$\Rightarrow J \cdot \mathfrak{a}_m \subset \overline{\mathcal{J}(\mathfrak{b}_\bullet^m)} = \mathcal{J}(\mathfrak{b}_\bullet^m)$ ▨

Symbolic powers

(Swanson [Sw]) $R$: normal domain

$R \supset \mathfrak{p}$: prime

$\exists\, k = k(\mathfrak{p}) \in \mathbb{N}$ s.t. $\mathfrak{p}^{(km)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$

Q. What is $k$ ?

Thm (Ein-Lazarsfeld-Smith [ELS])

$R$: regular affine domain $/\mathbb{C}$ (f.g. alg. over $\mathbb{C}$)

$\mathfrak{p} \subset R$: prime of ht. $h$

$\Rightarrow \mathfrak{p}^{(hm)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$ (i.e. $k(\mathfrak{p}) = \mathrm{ht}.\mathfrak{p}$)

Thm (Hochster-Huneke [HH1])

$R$: regular ring of equal char., $\mathfrak{p} \subset R$: prime of ht $h$.

$\mathfrak{p}^{(hm)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$

Singular case

Thm (T- [Ta2])

$R$: affine domain $/k$, $k$: perfect field of char. $p>0$

$\mathfrak{p} \subset R$: prime of ht. $h$, $J \subset R$: Jacobian ideal

$\Rightarrow J^{m-1}\tau(R)\mathfrak{p}^{(hm)} \subset \mathfrak{p}^m$, $\forall m \in \mathbb{N}$

<u>proof of ELS</u>

$\mathfrak{p}^{(\cdot)} := \{\mathfrak{p}^{(m)}\}_{m \in \mathbb{N}}$

$\mathfrak{p}^{(hm)} \subset \mathcal{J}(hm \cdot \mathfrak{p}^{(\cdot)}) \subset \mathcal{J}(h \cdot \mathfrak{p}^{(\cdot)})^m$

<u>ETS</u> $\mathcal{J}(h \cdot \mathfrak{p}^{(\cdot)}) \subset \mathfrak{p}$

$$\begin{aligned} \mathcal{J}(h \cdot \mathfrak{p}^{(\cdot)})_{\mathfrak{p}} &= \mathcal{J}(h \cdot \mathfrak{p}^{(\cdot)} R_{\mathfrak{p}}) \\ &= \mathcal{J}((\mathfrak{p} R_{\mathfrak{p}})^h) \subset \mathfrak{p} R_{\mathfrak{p}} \\ &\rightsquigarrow \mathcal{J}(h \cdot \mathfrak{p}^{(\cdot)}) \subset \mathfrak{p} \quad \blacksquare \end{aligned}$$

<u>proof of sing. case</u>

$\mathfrak{p}^{(\cdot)} := \{\mathfrak{p}^{(m)}\}_{m \in \mathbb{N}}$

$$\begin{aligned} J^{m-1} \mathcal{J}(R)\, \mathfrak{p}^{(hm)} &\subset \mathcal{J}(hm \cdot \mathfrak{p}^{(\cdot)})\, J^{m-1} \\ &\subset \underbrace{\mathcal{J}(h \cdot \mathfrak{p}^{(\cdot)})}_{\subset\, \mathfrak{p}}{}^m \quad \blacksquare \end{aligned}$$

If $\mathfrak{p}$ is special

Can we get a better bound ?

<u>Thm</u> (Hochster-Huneke [HH2], T-Yoshida [TY])

$R$ : regular ring of equal char.

$\mathfrak{p} \subset R$ : prime of ht $h \geq 2$

If $R/\mathfrak{p}$ is F-pure or of dense F-pure type (See [Hu])

$\Rightarrow \mathfrak{p}^{(hm-1)} \subset \mathfrak{p}^m \quad \forall m \in \mathbb{N}$

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